MATH QUIZ 9

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Average line segment length.

$$L = \frac{\int_U \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\, dx_1\, dx_2\, dy_1\, dy_2}{\int_U\, dx_1\, dx_2\, dy_1\, dy_2}$$

$$U = \{(x_1, x_2, y_1, y_2) \mid 0 \leq x_1, x_2, y_1, y_2 \leq 1\}$$

$$\int_U dx_1\, dx_2\, dy_1\, dy_2 = 1$$

Change of variables.

$$\begin{pmatrix} \bar{x} \\ x_2 \\ \bar{y} \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \equiv \varphi(x_1, x_2, y_1, y_2)$$

$$\int_{\varphi(U)} \sqrt{\bar{x}^2 + \bar{y}^2}\, d\bar{x}\, d\bar{y}\, dx_2\, dy_2 = \int_U \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} |\, d\varphi|\, dx_1\, dx_2\, dy_1\, dy_2$$

$$D\varphi = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|D\varphi| = 1$$

$$L = \int_U \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\, dx_1\, dx_2\, dy_1\, dy_2 = \frac{1}{|D\varphi|} \int_{\varphi(U)} \sqrt{\bar{x}^2 + \bar{y}^2}\, d\bar{x}\, d\bar{y}\, dx_2\, dy_2$$

Collapse x_2, y_2 integrals.

$$\varphi(U) = \left\{ (\bar{x}, x_2, \bar{y}, y_2) \left| \begin{array}{c} -1 \leq \bar{x}, \bar{y} \leq 1 \\ \max(0, -\bar{x}) \leq x_2 \leq \min(1, 1 - \bar{x}) \\ \max(0, -\bar{y}) \leq y_2 \leq \min(1, 1 - \bar{y}) \end{array} \right. \right\}$$

$$L = \int_{-1}^{1} \int_{-1}^{1} \left(\int_{\max(0, -\bar{x})}^{\min(1, 1 - \bar{x})} dx_2 \int_{\max(0, -\bar{y})}^{\min(1, 1 - \bar{y})} dy_2 \right) \sqrt{\bar{x}^2 + \bar{y}^2}\, d\bar{x}\, d\bar{y}$$

$$\int_{\max(0, -a)}^{\min(1, 1 - a)} dx = \begin{cases} \int_0^{1-a} dx & a > 0 \\ \int_0^1 dx & a = 0 \\ \int_{-a}^1 dx & a < 0 \end{cases} = \begin{cases} (1 - a) - 0 & a > 0 \\ 1 - 0 & a = 0 \\ (1) - (-a) & a < 0 \end{cases} = 1 - |a|$$

$$L = \int_{-1}^{1} \int_{-1}^{1} \sqrt{\bar{x}^2 + \bar{y}^2}(1 - |\bar{x}|)(1 - |\bar{y}|)\, d\bar{x}\, d\bar{y} = 4 \int_0^1 \int_0^1 \sqrt{\bar{x}^2 + \bar{y}^2}(1 - \bar{x})(1 - \bar{y})\, d\bar{x}\, d\bar{y}$$

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Trig substitution to get single integral.

$$\bar{y} = \bar{x}\tan u$$
$$u = \operatorname{atan}\left(\frac{\bar{y}}{\bar{x}}\right)$$
$$\bar{y} \in (0,1) \mapsto u \in \left(0, \operatorname{atan}\left(\frac{1}{\bar{x}}\right)\right)$$
$$d\bar{y} = \bar{x}\sec^2 u\, du$$
$$L = 4\int_0^1 \int_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} \sqrt{\bar{x}^2 + \bar{x}^2\tan^2 u}(1-\bar{x})(1-\bar{x}\tan u)\sec^2 u\, du\, d\bar{x}$$
$$= 4\int_0^1 (\bar{x}^2 - \bar{x}^3)\left(\int_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} \sec^3 u(1-\bar{x}\tan u)\, du\right) d\bar{x}$$

Integral breakdown.

$$L = 4\int_0^1 (\bar{x}^2 - \bar{x}^3)(\alpha(\bar{x}) - \beta(\bar{x})\bar{x})\, d\bar{x}$$

Secant reduction rule.

$$\alpha(\bar{x}) = \int_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} \sec^3 u\, du = \frac{1}{2}(\tan u \sec u|_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} + \int_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} \sec u\, du) = \frac{1}{2}(\frac{1}{\bar{x}} \cdot \frac{\sqrt{1+\bar{x}^2}}{\bar{x}} - 0\cdot 1 + \gamma(\bar{x}))$$
$$= \frac{\sqrt{1+\bar{x}^2}}{2\bar{x}^2} + \frac{1}{2}\ln(1+\sqrt{1+\bar{x}^2}) - \frac{1}{2}\ln\bar{x}$$

Integral of secant.

$$\gamma(\bar{x}) = \int_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} \sec u\, du = \ln(\tan u + \sec u)|_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} = \ln\left(\frac{1/\bar{x} + \sqrt{1+\bar{x}^2}/\bar{x}}{1+0}\right) = \ln(1+\sqrt{1+\bar{x}^2}) - \ln\bar{x}$$

(**Note.** $\sec(\operatorname{atan}(b/a)) = \frac{\sqrt{b^2+a^2}}{a}$ by inspecting a right triangle)

u substitution.

$$\beta(\bar{x}) = \int_0^{\operatorname{atan}\left(\frac{1}{\bar{x}}\right)} \sec^3 u \tan u\, du$$
$$v = \sec u$$
$$u \in \left(0, \operatorname{atan}\left(\frac{1}{\bar{x}}\right)\right) \mapsto v \in \left(1, \frac{\sqrt{1+\bar{x}^2}}{\bar{x}}\right)$$
$$dv = \sec u \tan u\, du$$
$$B(\bar{x}) = \int_1^{\frac{\sqrt{1+\bar{x}^2}}{\bar{x}}} v^2\, dv = \left.\frac{v^3}{3}\right|_1^{\frac{\sqrt{1+\bar{x}^2}}{\bar{x}}} = \frac{\sqrt{1+\bar{x}^2}^3}{3\bar{x}^3} - \frac{1}{3}$$

Combine terms.

$$L = 4\int_0^1 (\bar{x}^2 - \bar{x}^3)(\frac{\sqrt{1+\bar{x}^2}}{2\bar{x}^2} + \frac{1}{2}\ln(1+\sqrt{1+\bar{x}^2}) - \frac{1}{2}\ln\bar{x} - \left(\frac{\sqrt{1+\bar{x}^2}^3}{3\bar{x}^3} - \frac{1}{3}\right)\bar{x})\, d\bar{x}$$

Group terms.

$$L = \int_0^1 (1-\bar{x})(2\sqrt{1+\bar{x}^2} - \frac{4}{3}\sqrt{1+\bar{x}^2}^3)\, d\bar{x} + 2\int_0^1 (\bar{x}^2 - \bar{x}^3)(\ln(1+\sqrt{1+\bar{x}^2}) - \ln\bar{x})\, d\bar{x} + \frac{4}{3}\int_0^1 (\bar{x}^3 - \bar{x}^4)\, d\bar{x}$$

$$= A + B + C$$

Polynomial integral.

$$C = \frac{4}{3}\int_0^1 (\bar{x}^3 - \bar{x}^4)\, d\bar{x} = \frac{4}{3}\left(\frac{\bar{x}^4}{4} - \frac{\bar{x}^5}{5}\right)\Big|_0^1 = \frac{4}{3}\left(\frac{1}{4} - \frac{1}{5} - 0\right) = \frac{1}{15}$$

Integration by parts.

$$B = 2\int_0^1 (\bar{x}^2 - \bar{x}^3)(\ln(1+\sqrt{1+\bar{x}^2}) - \ln\bar{x})\, d\bar{x}$$

$$u = \ln(1+\sqrt{1+\bar{x}^2}) - \ln\bar{x} \qquad dv = (\bar{x}^2 - \bar{x}^3)\, d\bar{x}$$
$$du = \tfrac{d\bar{x}}{1+\sqrt{1+\bar{x}^2}}\tfrac{\bar{x}}{\sqrt{1+\bar{x}^2}} - \tfrac{d\bar{x}}{\bar{x}} \qquad v = \tfrac{\bar{x}^3}{3} - \tfrac{\bar{x}^4}{4}$$

$$B = 2\left((\ln(1+\sqrt{1+\bar{x}^2}) - \ln\bar{x})\left(\frac{\bar{x}^3}{3} - \frac{\bar{x}^4}{4}\right)\Big|_0^1 - \int_0^1 \left(\frac{\bar{x}^3}{3} - \frac{\bar{x}^4}{4}\right)\left(\frac{\bar{x}}{\sqrt{1+\bar{x}^2}+1+\bar{x}^2} - \frac{1}{\bar{x}}\right) d\bar{x}\right)$$

simplifying polynomial terms and the remaining integrand.

$$= 2\ln(1+\sqrt{2})\left(\frac{1}{3} - \frac{1}{4}\right) + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right) d\bar{x} - 2\int_0^1 \left(\frac{\bar{x}^3}{3} - \frac{\bar{x}^4}{4}\right)\left(\frac{\sqrt{1+\bar{x}^2} - 1 - \bar{x}^2}{1+\bar{x}^2-1-2\bar{x}^2-\bar{x}^4}\right)\bar{x}\, d\bar{x}$$

(Note. $\frac{1}{a+b} = \frac{a-b}{(a+b)(a-b)} = \frac{a-b}{a^2-b^2}$, neat trick to move the square root to the numerator)

$$= \frac{1}{6}\ln(1+\sqrt{2}) + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right) d\bar{x} - 2\int_0^1 \left(\frac{\bar{x}^3}{3} - \frac{\bar{x}^4}{4}\right)\left(\frac{\sqrt{1+\bar{x}^2} - 1 - \bar{x}^2}{1+\bar{x}^2}\frac{\bar{x}}{-\bar{x}^2}\right) d\bar{x}$$

$$= \frac{1}{6}\ln(1+\sqrt{2}) + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right) d\bar{x} + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right)\left(\frac{1}{\sqrt{1+\bar{x}^2}} - 1\right) d\bar{x}$$

$$= \frac{1}{6}\ln(1+\sqrt{2}) + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right) d\bar{x} - 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right) d\bar{x} + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right)\frac{1}{\sqrt{1+\bar{x}^2}}\, d\bar{x}$$

$$= \frac{1}{6}\ln(1+\sqrt{2}) + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right)\frac{1}{\sqrt{1+\bar{x}^2}}\, d\bar{x}$$

Remaining integral term has a square root, so it's regrouped with A, giving A' and B'.

$$B = B' + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right)\frac{1}{\sqrt{1+\bar{x}^2}}\, d\bar{x}$$

$$B' = \frac{1}{6}\ln(1+\sqrt{2})$$

A' terms.

$$A' = A + 2\int_0^1 \left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right)\frac{1}{\sqrt{1+\bar{x}^2}}\, d\bar{x}$$

$$A' = \int_0^1 \left((1-\bar{x})\left(2\sqrt{1+\bar{x}^2} - \frac{4}{3}\sqrt{1+\bar{x}^2}^3\right) + 2\left(\frac{\bar{x}^2}{3} - \frac{\bar{x}^3}{4}\right)\frac{1}{\sqrt{1+\bar{x}^2}}\right) d\bar{x}$$

Group by even/odd.

$$A' = \int_0^1 \left(2\sqrt{1+\bar{x}^2} - \frac{4}{3}\sqrt{1+\bar{x}^2}^3 + \frac{2}{3}\bar{x}^2\frac{1}{\sqrt{1+\bar{x}^2}}\right) d\bar{x} - \int_0^1 \left(2\bar{x}\sqrt{1+\bar{x}^2} - \frac{4}{3}\bar{x}\sqrt{1+\bar{x}^2}^3 + \frac{1}{2}\bar{x}^3\frac{1}{\sqrt{1+\bar{x}^2}}\right) d\bar{x}$$

$$= A_1 - A_2$$

u substitution on A_2.

$$A_2 = \int_0^1 \left(2\bar{x}\sqrt{1+\bar{x}^2} - \frac{4}{3}\bar{x}\sqrt{1+\bar{x}^2}^3 + \frac{1}{2}\bar{x}^3\frac{1}{\sqrt{1+\bar{x}^2}}\right)\, d\bar{x}$$

$$= \int_0^1 \left(2\bar{x}\sqrt{1+\bar{x}^2} - \frac{4}{3}\bar{x}\sqrt{1+\bar{x}^2}^3 + \frac{1}{2}\bar{x}\frac{(1+\bar{x}^2)-1}{\sqrt{1+\bar{x}^2}}\right)\, d\bar{x}$$

$$u = 1+\bar{x}^2$$

$$\bar{x} \in (0,1) \mapsto u \in (1,2)$$

$$du = 2\bar{x}\, d\bar{x}$$

$$A_2 = \int_1^2 \left(\sqrt{u} - \frac{2}{3}\sqrt{u}^3 + \frac{1}{4}\frac{u-1}{\sqrt{u}}\right)\, du = \int_1^2 \left(\frac{5}{4}\sqrt{u} - \frac{2}{3}\sqrt{u}^3 - \frac{1}{4}\frac{1}{\sqrt{u}}\right)\, du$$

$$= \frac{5}{6}\sqrt{u}^3 - \frac{4}{15}\sqrt{u}^5 - \frac{1}{2}\sqrt{u}\Big|_1^2 = \frac{5}{3}\sqrt{2} - \frac{16}{15}\sqrt{2} - \frac{1}{2}\sqrt{2} - \frac{5}{6} + \frac{4}{15} + \frac{1}{2} = \frac{1}{10}\sqrt{2} - \frac{1}{15}$$

Trig substitution on A_1.

$$A_1 = \int_0^1 \left(2\sqrt{1+\bar{x}^2} - \frac{4}{3}\sqrt{1+\bar{x}^2}^3 + \frac{2}{3}\bar{x}^2\frac{1}{\sqrt{1+\bar{x}^2}}\right)\, d\bar{x}$$

$$\bar{x} = \tan u$$

$$u = \operatorname{atan}\bar{x}$$

$$x \in (0,1) \mapsto u \in \left(0, \frac{\pi}{4}\right)$$

$$d\bar{x} = \sec^2 u\, du$$

$$A_1 = \int_0^{\frac{\pi}{4}} \left(2\sqrt{1+\tan^2 u} - \frac{4}{3}\sqrt{1+\tan^2 u}^3 + \frac{2}{3}\tan^2 u\frac{1}{\sqrt{1+\tan^2 u}}\right)\sec^2 u\, du$$

$$= \int_0^{\frac{\pi}{4}} \left(2\sec^3 u - \frac{4}{3}\sec^5 u + \frac{2}{3}(\sec^2 u - 1)\sec u\right)\, du$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{4}} \sec^3 u\, du - \frac{4}{3}\int_0^{\frac{\pi}{4}} \sec^5 u\, du - \frac{2}{3}\int_0^{\frac{\pi}{4}} \sec u\, du$$

Secant reduction rule.

$$= \frac{8}{3}\int_0^{\frac{\pi}{4}} \sec^3 u\, du - \frac{4}{3}\frac{1}{4}\left(\tan u\sec^3 u\Big|_0^{\frac{\pi}{4}} + 3\int_0^{\frac{\pi}{4}} \sec^3 u\, du\right) - \frac{2}{3}\int_0^{\frac{\pi}{4}} \sec u\, du$$

$$= \frac{5}{3}\int_0^{\frac{\pi}{4}} \sec^3 u\, du - \frac{1}{3}(1\cdot 2\sqrt{2} - 0\cdot 1) - \frac{2}{3}\int_0^{\frac{\pi}{4}} \sec u\, du$$

$$= \frac{5}{3}\frac{1}{2}\left(\tan u\sec u\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \sec u\, du\right) - \frac{2}{3}\sqrt{2} - \frac{2}{3}\int_0^{\frac{\pi}{4}} \sec u\, du$$

$$= \frac{5}{6}(1\cdot\sqrt{2} - 0\cdot 1) - \frac{2}{3}\sqrt{2} + \frac{1}{6}\int_0^{\frac{\pi}{4}} \sec u\, du$$

$$= \frac{5}{6}(1\cdot\sqrt{2} - 0\cdot 1) - \frac{2}{3}\sqrt{2} + \frac{1}{6}\ln(1+\sqrt{2})$$

$$= \frac{1}{6}\ln(1+\sqrt{2}) + \frac{1}{6}\sqrt{2}$$

Finally, collect all terms back together.

$$
\begin{aligned}
&L = A_1 - A_2 + B' + C \\
&= \frac{1}{6}\ln(1+\sqrt{2}) + \frac{1}{6}\sqrt{2} - \frac{1}{10}\sqrt{2} + \frac{1}{15} + \frac{1}{6}\ln(1+\sqrt{2}) + \frac{1}{15} \\
&= \frac{1}{3}\ln(1+\sqrt{2}) + \frac{1}{15}\sqrt{2} + \frac{2}{15} \\
&\approx 0.52140543316
\end{aligned}
$$

8 digits: 0.52140543

5 digits: 0.52141