Introduction

In this paper, I'm going to solve 12 problems:

> What is the average [square of the] length of a line segment within a unit [circle|sphere], with [neither|one|both] endpoints constrained to be on the boundary of the shape?

And, as a bit of lagniappe[†], at the end I shall solve this question proposed by Ether in a private message:

> What is the average of the lengths of all the chords of the unit ***semi*** circle?

I'm going to place language being understood by a non-mathematician over rigor. Where I'm providing additional clarification to the non-mathematician, I'll add comments in (parenthesis).

Mean length and square length of chords on the circle

As warmup, let's start with the two problems which involve a single integration. That is:

> What is the average [square of the] length of a line segment within a circle, with both endpoints constrained to be on the boundary of the shape?

In pedantic form, these problems would take the form of double integrals:

$$\int_0^{2\pi}\int_0^{2\pi} 2R|\sin((\theta_2-\theta_1)/2)|\, d\theta_2\, d\theta_1 \Big/ \int_0^{2\pi}\int_0^{2\pi} d\theta_2\, d\theta_1 \quad \text{and}$$

$$\int_0^{2\pi}\int_0^{2\pi} (2R|\sin((\theta_2-\theta_1)/2)|)^2\, d\theta_2\, d\theta_1 \Big/ \int_0^{2\pi}\int_0^{2\pi} d\theta_2\, d\theta_1$$

However, whenever feasible, I am going to simplify integrals by taking advantage of the abundant radial symmetry of the circle (and later, the even more abundant symmetry of the sphere). I'll be using this sort of symmetry except for the lagniappe question at the end, so please make sure you are comfortable with it before moving on.

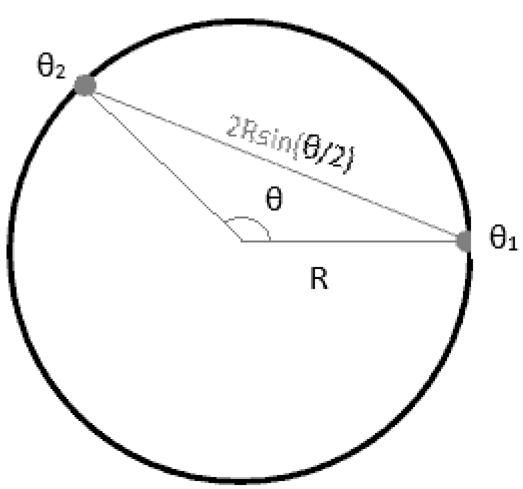

Figure 1: Chord Length

θ_1 and θ_2 do not individually does not matter, because of the radial symmetry. What matters is the angular separation between these two, which I shall designate as θ_2-θ_1, and which is always 0 or between 0 and π. To see this, center the circle at the origin of a polar coordinate system, and after selection of θ_1, rotate the circle so that θ_1 is at θ=0, as shown in figure 1 at left. If the result is negative, flip over the x axis (which does not change the chord length). θ = θ_2-θ_1 is readily seen to be evenly spaced around an angle of [0,π]. The length of the chord from θ_2 to θ_1 is seen to be 2Rsin(θ/2).

The average arc length is now:

$$\int_0^{\pi} 2R\sin(\theta/2)\, d\theta \Big/ \int_0^{\pi} d\theta = 2R/\pi \int_0^{\pi} \sin(\theta/2)\, d\theta$$

letting $\alpha = \theta/2$, $d\theta = 2d\alpha$, this becomes:

$$2R/\pi \int_0^{\pi/2} \sin\alpha \, 2 \, d\alpha = 4R/\pi \Big[-\cos\alpha \Big]_0^{\pi/2} = 4R/\pi \Big[0 + 1 \Big] = 4R/\pi$$

Likewise, the average square of the arc length is:

$$\int_0^{\pi} (2R\sin(\theta/2))^2 \, d\theta \Big/ \int_0^{\pi} d\theta = 4R^2/\pi \int_0^{\pi} \sin^2(\theta/2) \, d\theta$$

Making use of a half angle formula rather than a change of variables:

$$4R^2/\pi \int_0^{\pi} (1-\cos\theta)/2 \, d\theta = 2R^2/\pi \Big[\theta - \sin\theta \Big]_0^{\pi} = 2R^2/\pi \Big[\pi - 0 - 0 + 0 \Big] = 2R^2$$

Mean length and square length of line segments with one end on the circle

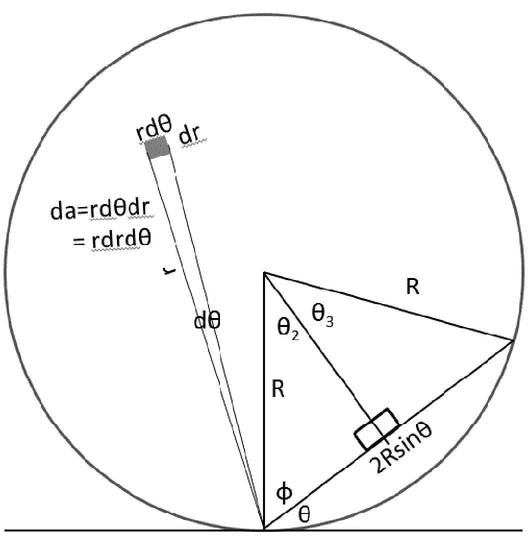

Figure 2: Radial Coordinates

Here, we will begin by selecting the point with θ_1 as its coordinate to be constrained to be on the circle. We shall then move the circle so that the constrained point is at the origin of a polar coordinate system, with the center of the circle along the radial $\theta=\pi/2$, as shown in Figure 2.

The circle (that is, the border) shown can be described in this coordinate system by the equation $r = 2R\sin\theta$. This can be seen to be the case by noticing that the two angles at the origin sum to a right angle, so $\phi = \pi/2 - \theta$. As the angle at the small square is a right angle by construction, and the sum of the angles on the lower triangle are π (180º), the angle at the center of the circle (θ_2) on this triangle also has value θ. Because the large triangle is isosceles, the other angle on the circle is ϕ and the other half of the central angle (θ_3) also has value θ. As the radius is R, the length of each half-chord is $R\sin\theta$, so the length of the chord is $2R\sin\theta$ at every point for $0 <= \theta < = \pi$.

The average length of a line segment is found by integrating r for each point inside the circle (weighted by area), and dividing by the area of the circle. We will refer to the numerator as Ř and the denominator as Ă:

$$\int^{\bullet} r\,da \Big/ \int^{\bullet} da = \check{R}/\breve{A}$$

In the cartesian (right angle) coordinate system, da is simply dx dy (or dy dx). In polar coordinates, da is well known as rdrdθ. If you are not familiar with this, see figure 2 as an indication that an infinitesimally small annular segment approaches a rectangle of area rdrdθ. Also, follow the integration of the area in the next paragraph for a bit of sanity check.

$$\check{R}/\check{A} = \int_0^{\pi}\int_0^{2R\sin\theta} r^2 dr d\theta \Big/ \int_0^{\pi}\int_0^{2R\sin\theta} r dr d\theta$$

In the interests of checking work and comforting those not comfortable with polar coordinates, let's first evaluate Ă, which should equal the area of the circle of radius R.

$$\check{A} = \int_0^{\pi}\int_0^{2R\sin\theta} r dr d\theta = \int_0^{\pi}\left[r^2/2\right]_0^{2R\sin\theta} d\theta = \int_0^{\pi} 2R^2\sin^2\theta\, d\theta = 2R^2\int_0^{\pi} \sin^2\theta\, d\theta$$

The integral of $\sin^2\theta$ can be solved using the half-angle formula as for the preceding problem, or looked up in a table of integrals, either way giving:

$$\check{A} = 2R^2\int_0^{\pi} \sin^2\theta\, d\theta = 2R^2\left[\theta/2 - ¼\sin(2\theta)\right]_0^{\pi} = 2R^2\left[\pi/2 - 0 – 0 + 0\right] = \pi R^2$$

This clearly agrees with the well-known formula for the area of a circle. Similarly,

$$\check{R} = \int_0^{\pi}\int_0^{2R\sin\theta} r^2 dr d\theta = \int_0^{\pi}\left[r^3/3\right]_0^{2R\sin\theta} d\theta = \int_0^{\pi} 8R^3\sin^3\theta/3\, d\theta = 8R^3/3\int_0^{\pi} \sin^3\theta\, d\theta$$

The integral of $\sin^3\theta$ is easily found by substituting $\sin^2\theta = 1- \cos^2\theta$:

$$\check{R} = 8R^3/3\int_0^{\pi} \sin\theta-\cos^2\theta\sin\theta d\theta = 8R^3/3\left[-\cos\theta+\cos^3\theta/3\right]_0^{\pi} = 8R^3/3\left[1+1-⅓-⅓\right] = 32R^3/9$$

The mean length of these segments with one end on the circle is then:

$$\check{R}/\check{A} = 32R^3/9 / \pi R^2 = 32R/9\pi$$

The mean square of the length of these segments with one end on the circle is:

$$\int\bullet r^2 da / \int\bullet da = \check{S}/\check{A}$$

$\check{A} = \pi R^2$ as before, and we can evaluate Š similarly to both Ř and Ă:

$$\check{S} = \int_0^{\pi}\int_0^{2R\sin\theta} r^3 dr d\theta = \int_0^{\pi}\left[r^4/4\right]_0^{2R\sin\theta} d\theta = \int_0^{\pi} 4R^4\sin^4\theta\, d\theta = 4R^4\int_0^{\pi} \sin^4\theta\, d\theta$$

The integral of $\sin^4\theta$ is found by substituting $\sin^2\theta = 1- \cos^2\theta$ and using a half angle formula:

$$\check{S} = 4R^4\int_0^{\pi} \sin^2\theta - \cos^2\theta\sin^2\theta\, d\theta = 4R^4\int_0^{\pi} \sin^2\theta - ¼\sin^2 2\theta\, d\theta$$

The first term of the integral was previously calculated during the solution for Ă, yielding $\pi/2$. W solve the second through substituting $\alpha = 2\theta$, so that $d\theta = ½d\alpha$:

$$Š = 2\pi R^4 - R^4 \int_0^{2\pi} \sin^2\alpha \, ½d\alpha = 2\pi R^4 - ½R^4 \int_0^{2\pi} \sin^2\alpha \, d\alpha$$

We are once again integrating $\sin^2$, and use the same formula:

$$Š = 2\pi R^4 - ½R^4 \Big[\alpha/2 - ¼\sin(2\alpha)\Big]_0^{2\pi} = 2\pi R^4 - ½R^4 \Big[\pi - 0 - 0 + 0\Big] = 3\pi R^4/2$$

And the mean square length is:

$$Š/Ă = 3\pi R^4/2 \;/\; \pi R^2 = 3R^2/2$$

Mean length and square length of line segments with both ends within the circle

The form of the averages above (with one end on the circle boundary) can be used to evaluate the average length of any segment, if we can determine the population density Ᵽ(r) of the larger of the radii of two randomly chosen points; if we scale Ᵽ(r) so that the total population is one, we integrate Ᵽ(r) times the average length of lines going from the border to within the smaller circle.

As a stepping-stone to this population density, consider the question: If two points are randomly chosen within the unit circle (equal area is equal probability), what is the likelihood that BOTH of them have an r coordinate which is less than or equal to R (for 1 >= R >= 0)?
A: The chances of the first point being no greater than R are the area inside a circle of size R divided by the area of the unit circle: $\pi R^2 / \pi\, 1^2 = R^2$. The chances of the second point being no greater than R are also R^2. The chances of BOTH points being within R of the center is therefore $R^2R^2 = R^4$.
That is,

$$\int_0^R Ᵽ(r)\, dr = R^4$$

and we find $Ᵽ(r) = 4r^3$.

The average length of a line segment within the unit circle is therefore:

$$\int_0^1 4r^3 \bullet 32r/9\pi \, dr = 128/9\pi \int_0^1 r^4 \, dr = 128/9\pi \Big[r^5/5\Big]_0^1 = 128/45\pi$$

Similarly the average square of the length of a line segment within the unit circle is:

$$\int_0^1 4r^3 \bullet 3r^2/2 \, dr = 6 \int_0^1 r^5 \, dr = 6\Big[r^6/6\Big]_0^1 = 1$$

Spherical problems

The spherical cases can all be solved similarly, most conveniently in spherical coordinates. The spherical coordinates are defined:

ρ = distance from the origin = $\sqrt{(x^2+y^2+z^2)}$

ϕ = angle from the north pole/z axis, aka colatitude = $\cos^{-1}(z/\rho)$

θ = azimuthal angle, aka longitude = atan2(y,x)

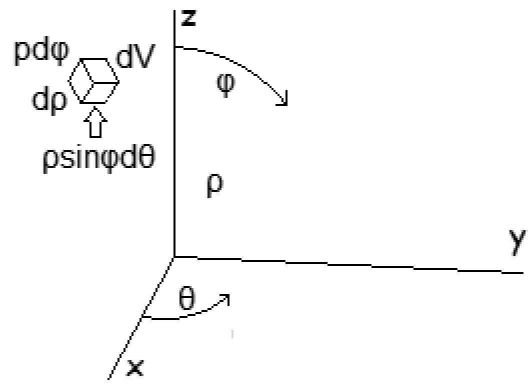

Figure 3: Spherical Coordinates

Converting back, we would use:

$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi$

As we shall be doing an integral over the spherical volume, we must examine a small volume to find that:

$dV = d\rho \quad \rho d\phi \quad \rho \sin\phi d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Mean length and square length of line segments with both ends on the sphere

For this problem, I shall place the center of the sphere at the origin of the spherical coordinate system, and take advantage of spherical symmetry to rotate it so that the "first point" is at ϕ=0 (the north pole). (**Not** as shown in Figure 3.) As with the chords on the circle, the other endpoint is equally distributed around the sphere. The length of the chord is calculated much as in the circular case, and (referring back to figure 1, inverting the figure, and replacing θ with ϕ) is seen to be $2R\sin(\phi/2)$. Since we are only integrating on a surface of constant ρ =R, the integrals only have two dimensions; these further reduce to two single dimension integrals because there are no inter-dependencies between θ and ϕ:

$$\check{R}/\breve{A} = \int^{\bullet} \lambda da \Big/ \int^{\bullet} da \qquad \check{S}/\breve{A} = \int^{\bullet} \lambda^2 da \Big/ \int^{\bullet} da$$

Solving first for $\breve{A}$, the area of the sphere:

$$\breve{A} = \int^{\bullet} da = \int_0^{2\pi}\int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\theta = \rho^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = \rho^2 \, 2\pi \Big[-\cos\phi\Big]_0^{\pi} = 4\pi\rho^2$$

This agrees with the well-known formula. Solving next for $\check{R}$:

$$\check{R} = \int^{\bullet} \lambda da = \int_0^{2\pi}\int_0^{\pi} 2\rho\sin(\phi/2) \, \rho^2 \sin\phi \, d\phi \, d\theta = 4\pi\rho^3 \int_0^{\pi} \sin(\phi/2) \sin\phi \, d\phi$$

Substituting $\alpha = \phi/2$, $\sin\phi = \sin 2\alpha = 2 \sin\alpha \cos\alpha$, and $d\phi = 2d\alpha$:

$$Ř = 4\pi\rho^3 \int_0^{\pi/2} \sin\alpha \;\; 2 \sin\alpha \cos\alpha \;\; 2d\alpha = 16\pi\rho^3 \int_0^{\pi/2} \sin^2\alpha \cos\alpha d\alpha$$

$$Ř = 16\pi\rho^3 \Big[⅓\sin^3\alpha \Big]_0^{\pi/2} = 16\pi\rho^3 \Big[⅓ - 0 \Big] = 16\pi\rho^3/3$$

Similarly for Š:

$$Š = \int\!\bullet \lambda^2 da = \int_0^{2\pi}\int_0^{\pi} (2\rho\sin(\phi/2))^2 \; \rho^2 \sin\phi \; d\phi \; d\theta = 8\pi\rho^4 \int_0^{\pi} \sin^2(\phi/2) \sin\phi \; d\phi$$

$$Š = 8\pi\rho^4 \int_0^{\pi/2} \sin^2\alpha \;\; 2 \sin\alpha \cos\alpha \;\; 2d\alpha = 32\pi\rho^4 \int_0^{\pi/2} \sin^3\alpha \cos\alpha d\alpha$$

$$Š = 32\pi\rho^4 \Big[¼\sin^4\alpha \Big]_0^{\pi/2} = 32\pi\rho^4 \Big[¼ - 0 \Big] = 8\pi\rho^4$$

The mean length and mean square length of chords on the unit sphere are therefore:

$$Ř/Ă = 16\pi\rho^3/3 \Big/ 4\pi\rho^2 = 4\rho/3 = 4/3 \qquad\qquad Š/Ă = 8\pi\rho^4 \Big/ 4\pi\rho^2 = 2\rho^2 = 2$$

Mean length and square length of line segments with one end on the sphere

For the cases where one or both of the endpoints is constrained to be on the sphere, we rotate and translate the sphere so that that endpoint is at the origin of a spherical coordinate system, with the center of the sphere along the ϕ=0 axis (north pole). The mean length and square length are:

$$Ř/\tilde{V} = \int\!\bullet \rho dV \Big/ \int\!\bullet dV \qquad Š/\tilde{V} = \int\!\bullet \rho^2 dV \Big/ \int\!\bullet dV$$

Again, we'll solve the common denominator Ṽ, which is the volume of the sphere, as a sanity check. Note that to limit the integration to the volume, we limit ρ to 2Rcosϕ. This is much like the circular case, though using cosϕ instead of sinθ because ϕ corresponds to $\pi/2 - \theta_{circle}$. We limit integration on ϕ to π/2, because this sphere is entirely above the z=0 plane.

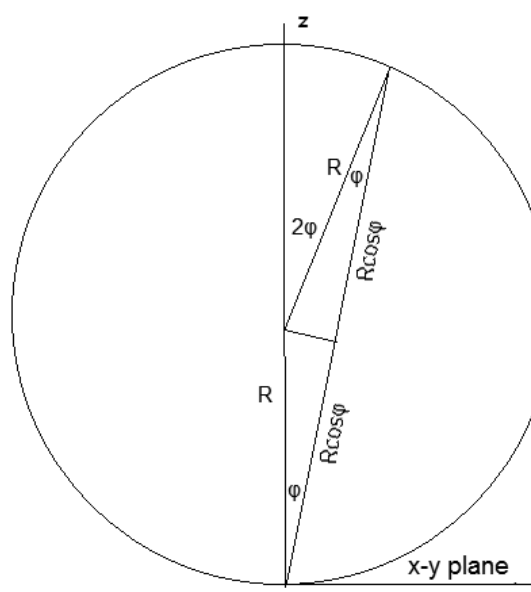

Figure 4: Spherical Chord Length

$$\tilde{V} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Because θ appears nowhere except in the dθ, we can again separate this integral from the others:

$$\tilde{V} = \int_0^{2\pi} d\theta \int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi = 2\pi\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi$$

The integral over ρ is a simple power, and the integral over ϕ that results is recognized as a derivative of $\cos^n\phi$:

$$\tilde{V} = 2\pi\int_0^{\pi/2} \Big[\rho^3/3\Big]_0^{2R\cos\phi} \sin\phi \, d\phi = 16\pi R^3/3\int_0^{\pi/2} \cos^3\phi\sin\phi \, d\phi$$

$$\tilde{V} = 16\pi R^3/3\Big[-\cos^4\phi \,/\, 4\Big]_0^{\pi/2} = 16\pi R^3/3\Big[0 + 1 \,/\, 4\Big] = 4\pi R^3/3$$

Which checks with the classic solution for the volume of a sphere. Similarly:

$$\check{R} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^3\sin\phi \, d\rho \, d\phi$$

$$\check{R} = 2\pi\int_0^{\pi/2} \Big[\rho^4/4\Big]_0^{2R\cos\phi} \sin\phi \, d\phi = 8\pi R^4\int_0^{\pi/2} \cos^4\phi\sin\phi \, d\phi$$

$$\check{R} = 8\pi R^4\Big[-\cos^5\phi \,/\, 5\Big]_0^{\pi/2} = 8\pi R^4\Big[0 + 1 \,/\, 5\Big] = 8\pi R^4/5$$

And:

$$\check{S} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^4\sin\phi \, d\rho \, d\phi$$

$$\check{S} = 2\pi\int_0^{\pi/2} \Big[\rho^5/5\Big]_0^{2R\cos\phi} \sin\phi \, d\phi = 64\pi R^5/5\int_0^{\pi/2} \cos^5\phi\sin\phi \, d\phi$$

$$\check{S} = 64\pi R^5/5\Big[-\cos^6\phi \,/\, 6\Big]_0^{\pi/2} = 64\pi R^5/5\Big[0 + 1 \,/\, 6\Big] = 32\pi R^5/15$$

The mean length and mean square for segments with one end on the sphere are therefore:

$$\check{R}/\tilde{V} = 8\pi R^4/5 \Big/ 4\pi R^3/3 = 6R/5 \qquad \check{S}/\tilde{V} = 32\pi R^5/15 \Big/ 4\pi R^3/3 = 8R^2/5$$

Mean length and square length of line segments within the sphere

As with the circular case, we shall use the results above and integrate these over the population density of the outermost point as a function of ρ. The chances of a point on a circle having r < R was shown to be equal to R^2; by the same argument, the chances of a point in a sphere having ρ < R is R^3. The chances of both being less than R is $R^3 R^{3=} R^6$. The population density then becomes:

$$\int_0^R Ᵽ(\rho)\, d\rho = R^6 \;\Rightarrow\; Ᵽ(\rho) = 6\rho^5$$

The average length of a line segment within the unit sphere is therefore:

$$\int_0^1 6\rho^5 \bullet 6\rho/5\, d\rho = 36/5\int_0^1 \rho^6\, d\rho = 36/5\Big[\rho^7/7\Big]_0^1 = 36/35$$

Similarly the average square of the length of a line segment within the unit sphere is:

$$\int_0^1 6\rho^5 \bullet 8\,\rho^2/5\, d\rho = 48/5\int_0^1 \rho^7\, d\rho = 48/5\Big[\rho^8/8\Big]_0^1 = 6/5$$

Lagniappe: What is the average of the lengths of all the chords of the unit *semi* circle?

The points are selected along one degree of freedom, and the distance is a function of the separation between the points, so we have:

$$Ⱡ = \int Ᵽ(\lambda)\, \Lambda(\lambda)\, d\lambda,$$

where $\Lambda(\lambda)$ is the length function and $Ᵽ(\lambda)$ is the population density, normalized so that $\int Ᵽ(\lambda)d\lambda = 1$.

For convenience in the form of $\Lambda(\lambda)$, I select $\lambda = |\theta_2-\theta_1|/2$, so that $\Lambda(\lambda) = 2R\sin\lambda$, or simply $2\sin\lambda$ as this is the unit circle. λ has a domain from 0 to π/2.
By the same argument as for each dimension in the original square problem, we find that $Ᵽ(\lambda)$ scales linearly from a maximum at λ =0 down to zero at λ = π/2. We select scaling so that the integral over 0 to π is 1, and evaluate $Ᵽ(\lambda) = 8(\pi/2-\lambda)/\pi^2$. The average length of the chords of the unit semi-circle is then:

$$Ⱡ = \int Ᵽ(\lambda)\, \Lambda(\lambda)\, d\lambda = \int_0^{\pi/2} 8(\pi/2-\lambda)/\pi^2\; 2\sin\lambda\, d\lambda = 16/\pi^2 \int_0^{\pi/2} (\pi/2-\lambda)\sin\lambda\, d\lambda$$

$$Ⱡ = 16/\pi^2 \int_0^{\pi/2} \pi/2\,\sin\lambda - \lambda\sin\lambda\, d\lambda = 16/\pi^2 \Big[-\pi/2\cos\lambda + \lambda\cos\lambda - \sin\lambda\Big]_0^{\pi/2}$$

$$Ł = 16/\pi^2 \left[\pi/2 - 0 - 0 + 0 - 1 + 0 \right] = 8/\pi - 16/\pi^2, \text{ or } (8\pi - 16)/\pi^2$$

The squares of the lengths requires a few more integration tricks:

$$Š = \int Ᵽ(\lambda)\, \Lambda^2(\lambda)\, d\lambda = \int_0^{\pi/2} 8(\pi/2-\lambda)/\pi^2\, (2\sin\lambda)^2\, d\lambda = 32/\pi^2 \int_0^{\pi/2} (\pi/2-\lambda) \sin^2\lambda\, d\lambda$$

Substituting $\sin^2\lambda = (1-\cos 2\lambda)/2$, we have:

$$Š = 16/\pi^2 \int_0^{\pi/2} (\pi/2-\lambda)\,(1-\cos 2\lambda)\, d\lambda = 16/\pi^2 \int_0^{\pi/2} \pi/2-\lambda - \pi/2\cos 2\lambda + \lambda\cos 2\lambda\, d\lambda$$

Separating the polynomial term, and substituting $\alpha=2\lambda$, $\lambda=½\alpha$, $d\lambda=½d\alpha$ for the trig terms:

$$Š = 16/\pi^2 \int_0^{\pi/2} \pi/2-\lambda\ \ d\lambda\ +\ 16/\pi^2 \int_0^{\pi} (½\alpha \cos\alpha - \pi/2\cos\alpha)\ ½d\alpha$$

$$Š = 16/\pi^2 \left[\pi/2\lambda - ½\lambda^2\right]_0^{\pi/2} + 4/\pi^2 \int_0^{\pi} \alpha\cos\alpha - \pi\cos\alpha\, d\alpha$$

$$Š = 16/\pi^2 \left[\pi^2/4 - 0 - \pi^2/8 + 0\right] + 4/\pi^2 \left[\alpha\sin\alpha + \cos\alpha - \pi\sin\alpha\right]_0^{\pi}$$

$$Š = 2 + 4/\pi^2 \left[0 - 0\ - 1 - 1 - 0 + 0\ \right]\ = 2 - 8/\pi^2$$

Recap:

Segment Definition \ Dimension	Length (symbolic)	Length (numeric)	Square (symbolic)	Square (numeric)
Chords on circle	$4/\pi$	1.2732395	2	2.0000000
Segments with one end on circle	$32/9\pi$	1.1317685	3/2	1.5000000
Segments within circle	$128/45\pi$	0.9054148	1	1.0000000
Chords on sphere	4/3	1.3333333	2	2.0000000
Segments with one end on sphere	6/5	1.2000000	8/5	1.6000000
Segments within sphere	36/35	1.0285714	6/5	1.2000000
Chords on unit semi-circle	$8/\pi - 16/\pi^2$	0.9253402	$2 - 8/\pi^2$	1.1894305

† Yes, lagniappe is a real word, at least in this part of the world. It's pronounced LAN-yap, and means a bonus or unexpected gift, particularly one tacked on to a sale or contract. It also carries connotations of Deming's definition of quality: "exceeding expectations".