Shooting for FIRST

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1 Introduction

The problem of shooting a ball into a vertical target hole is examined for the 2006 FIRST robotics competition. The analyses which follow use the ideal projectile approximations without taking into account any drag or lift effects associated with interactions between the ball and the air. The results will hopefully be useful for game strategy, and for design requirements.

2 Coordinate System

For the following analyses of shooting a ball through a vertical target hole, the $+X$ direction is defined as pointing along the direction of the sidelines, into the playing court, and the $+Z$ direction is defined as upward from the ground. The $+Y$ direction is that to complete an XYZ right-handed coordinate system. The origin is defined as the projection of the center of the target hole on the ground below. Furthermore, in some calculations it is convenient to combine X and Y into a horizontal radial direction denoted as the R direction.

3 Ideal Kinematics

In the ideal world without drag due to air resistance, and with a constant gravitational force, horizontal acceleration is 0 and vertical acceleration is constant at $-g$. After integrating twice,

$$v = \int a\ dt \quad , \quad d = \int v\ dt \tag{1}$$

one obtains

$$\begin{aligned} d_r &= d_{ro} + v_{ro}t & (2) \\ v_z &= v_{zo} + a_{zo}t & (3) \\ d_z &= d_{zo} + v_{zo}t + \frac{1}{2}a_{zo}t^2 & (4) \end{aligned}$$

where the subscripts r and z indicate motion along the horizontal and vertical directions, respectively. For the following, we wish to calculate the kinematics of the motion corresponding to a launch angle θ with respect to the horizon. Suppose further that we are faced with the following boundary conditions:

$$\begin{aligned} d_{ro} &= 0 \\ v_{ro} &= v_o\cos(\theta) \\ v_{zo} &= v_o\sin(\theta) \\ a_{zo} &= -g \end{aligned} \tag{5}$$

Then (2) leads to:

$$d_r = v_o\cos(\theta)t_f \tag{6}$$

and (4) leads to:

$$\frac{1}{2}g\ {t_f}^2 - v_o\sin(\theta)t_f + (d_z - d_{zo}) = 0 \tag{7}$$

both at $t = t_f$, which together indicate when the ball arrives at the target. Solving for t_f:

$$t_f = \frac{v_o\sin(\theta) \pm \sqrt{{v_o}^2\sin^2(\theta) - 2g(d_z - d_{zo})}}{g} \tag{8}$$

Note that there are 2 solutions associated with the parabolic path of the ball passing through the height of d_z. The first solution is associated with a rising ball, while the second with a falling one. For a given launch angle θ, the discriminant under the radical helps identify the separation in *time* between the rising and falling portions of the parabolic path. So driving the discriminant to 0 by choosing $\theta \approx 22.4°$, would cause the vertex of the parabolic path to be at the desired height d_z. Note that since θ is the only variable determining t_f, one would not have the liberty of driving *both* the vertex of the parabolic path to height d_z, *and* choosing the horizontal range d_r.

4 Shooting Range

The maximum range can be found by searching for an extreme point in the graph of d_r versus θ. This can be done by taking the first derivative and setting to 0:

$$d_{r,max} = \{d_r |\ d(d_r)/d\theta = 0\} \tag{9}$$

Abbreviating the radical and its discriminant in (8) as:

$$\surd \equiv \sqrt{{v_o}^2 \sin^2(\theta) - 2g(d_z - d_{zo})} \tag{10}$$

and $\cos(\theta)$ and $\sin(\theta)$ as $c\theta$ and $s\theta$, the maximization relationship becomes

$$\begin{aligned} 0 = d(d_r)/d\theta &= \frac{v_o c\theta}{g}(v_o c\theta \pm \frac{{v_o}^2 s\theta c\theta}{\surd}) \\ &- \frac{v_o s\theta}{g}(v_o s\theta \pm \surd) \end{aligned} \tag{11}$$

which reduces to

$$c\theta(v_o c\theta \pm \frac{{v_o}^2 s\theta c\theta}{\surd}) = s\theta(v_o s\theta \pm \surd) \tag{12}$$

$$v_o c^2\theta \surd \pm {v_o}^2 s\theta c^2\theta = v_o s^2\theta \surd \pm s\theta \surd^2 \tag{13}$$

$$\pm s\theta \surd^2 + v_o(s^2\theta - c^2\theta)\surd \mp {v_o}^2 s\theta c^2\theta = 0 \tag{14}$$

In order for further simplify this, the $\surd$ term is treated as if it is a variable in a quadratic equation:

$$\surd = \frac{-v_o(s^2\theta - c^2\theta) \pm \sqrt{{v_o}^2(s^2\theta - c^2\theta)^2 + 4{v_o}^2 s^2\theta c^2\theta}}{\pm 2s\theta} \tag{15}$$

Substituting from (10):

$$\begin{gathered} \pm 2s\theta\sqrt{{v_o}^2 s^2\theta - 2g(d_z - d_{zo})} \\ = \\ -v_o(s^2\theta - c^2\theta) \pm \sqrt{{v_o}^2(s^2\theta - c^2\theta)^2 + 4{v_o}^2 s^2\theta c^2\theta} \end{gathered} \tag{16}$$

Squaring both sides results in:

$$\begin{gathered} 4s^2\theta({v_o}^2 s^2\theta - 2g(d_z - d_{zo})) \\ = \\ {v_o}^2(s^2\theta - c^2\theta)^2 \\ \mp 2v_o(s^2\theta - c^2\theta)\sqrt{{v_o}^2(s^2\theta - c^2\theta)^2 + 4{v_o}^2 s^2\theta c^2\theta} \\ + {v_o}^2(s^2\theta - c^2\theta)^2 + 4{v_o}^2 s^2\theta c^2\theta \end{gathered} \tag{17}$$

Combining terms on the RHS, moving some to the LHS, and simplifying the expression under the radical gives:

$$\begin{gathered} 2{v_o}^2(s^4\theta - c^4\theta) - 8g(d_z - d_{zo})s^2\theta \\ = \\ \mp 2{v_o}^2(s^2\theta - c^2\theta)\sqrt{(s^2\theta + c^2\theta)^2} \end{gathered} \tag{18}$$

This further simplifies to:

$$2{v_o}^2(s^4\theta - c^4\theta)(1 \pm 1) - 8g(d_z - d_{zo})s^2\theta = 0 \tag{19}$$

A trivial solution of $\theta = 0$ is obtained when the $(-)$ option of the $\pm$ is used, and for the $(+)$ option, the expression to solve becomes:

$${v_o}^2(s^4\theta - c^4\theta) - 2g(d_z - d_{zo})s^2\theta = 0 \tag{20}$$

Note that for the special case of $d_z = d_{zo}$, the result is $s\theta = \pm c\theta$, which leads to the well-known solution $\theta = 45°$ within the constraint $0 < \theta < 90°$.

The following example obtains the maximum range of an ideal 2006 FIRST competition ball, assuming that it is launched from the top of a 5 ft tall launcher:

$$\begin{aligned} & d_{zo} = 5 \text{ ft} \\ & d_z = 8.5 \text{ ft} \\ & v_o = (12 \text{ m/s})/(0.3048 \text{ m/ft}) \approx 39.37 \text{ ft/s} \\ & g = 32.17\ ft/s^2 \end{aligned} \tag{21}$$

Solving (20) numerically results in the following **maximum horizontal range** result:

$$\begin{aligned} \theta &\approx 47.25° \\ t_f &\approx 1.667 \text{ s} \\ d_r &\approx 44.54 \text{ ft} \end{aligned} \tag{22}$$

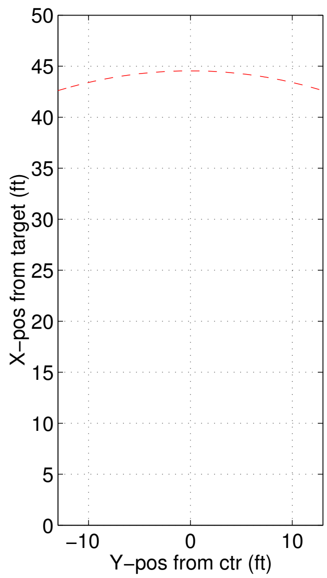

Figure 1: Shown is the line of maximum shooting range on the court.

Since the width of the court is 26 ft, i.e. distance between sidelines and the centerline is 13 ft, when the robot is against the side of the court, (assuming a point robot), using the Pythagorean theorem, the furthest the robot can be along the length of the court is

$$\sqrt{(44.54 \text{ ft})^2 - (13 \text{ ft})^2} \approx 42.61 \text{ ft} \tag{23}$$

The arc that represents the maximum range of the launcher is drawn using a dashed curve in Figure 1.

5 Launch Angle: $\theta = f(d_r, v_o)$

Another practical question is, assuming the launch speed is fixed, given the distance to the target, what should the launch angle θ be? Combining (6) and (7), one obtains

$$\frac{g}{2}\left(\frac{d_r}{v_o\cos(\theta)}\right)^2 - v_o\sin(\theta)\left(\frac{d_r}{v_o\cos(\theta)}\right) + (d_z - d_{zo}) = 0 \tag{24}$$

$$\frac{g{d_r}^2}{2{v_o}^2} - d_r\sin(\theta)\cos(\theta) + (d_z - d_{zo})\cos^2(\theta) = 0 \tag{25}$$

If $d_r = 0$, $v_o = 0$, or $(d_z - d_{zo}) = 0$, then θ can be obtained analytically. Otherwise, the solution θ must be obtained numerically.

Note that there are 2 solutions for θ. However this time the quadratic variable is not time, it is the launch angle. Therefore, the lower one is associated with the shallower launch angle, while the upper one with the steeper one. An example pair of parabolic paths is illustrated in Figure 2.

The problem of θ as a function of d_r has been solved and plotted in Figure 3. First, note that the lower curve starts high, reaches its minimum at $\theta \approx 22.4°$ (when the discriminant in (8) vanishes), and increases again. Physically since the height of the target is higher than that of the launcher, when the robot is very close to the target, it is aiming upward. Then as it moves away its aim lowers until the target is at the vertex of the parabolic path of the ball. Then the launch angle has to increase again. Note that for all of these there is a corresponding parabolic path which starts at a greater launch angle. These are shown by the upper curve. As the robot moves further away from the target, these two curves converge until they meet at $\theta \approx 47.25°$ which defines the maximum horizontal range of $d_r \approx 44.54$ ft, past which there is no solution.

6 The Rise and Fall

In order to maximize scoring in a fixed period of time, the ball should be delivered to the target as quickly as possible. Furthermore, if the air resistance is taken into account, then the ball is faster in the horizontal direction during the rise than during the fall. This would lead us to believe that the angle that the path of the ball makes with the vertical target is more perpendicular when rising than when falling. Therefore, the cross-section of the target's hole is larger during the rise, making it an easier target. For these reasons it's more prudent to score with a rising ball whenever possible.

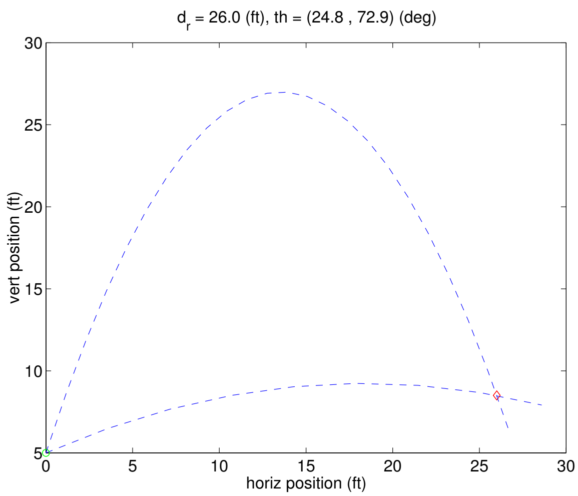

Figure 2: Shown are the lower and upper parabolic paths for a single horizontal range value to the target. Note that in both cases, the initial launch speed v_o is the same.

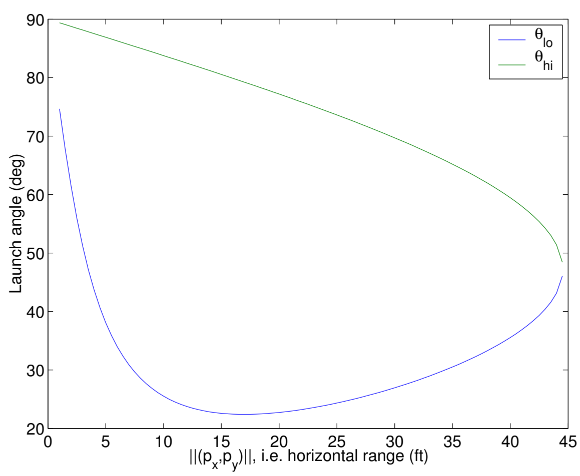

Figure 3: Shown are the lower and upper solutions for the launch angle θ as a function of horizontal distance d_r.

7 Clearance

As the ball is approaching the target hole, the size of the aperture experienced by the ball is different depending on the approach angle of the ball. Let us denote the *entry plane* as the plane perpendicular to the ball's motion as it enters the target hole. Then the clearance ΔD, is the smallest gap between opposite edges of the hoop of diameter D_h, and the ball of diameter D_b, projected onto the entry plane, as the ball enters it. This is calculated as:

$$\Delta D = D_h \cos(\phi) - D_b \tag{26}$$

where ϕ is the angle between the entry plane and the plane of the target's backboard. In order to compute ϕ, consider its horizontal component ϕ_r:

$$\phi_r = atan(p_y/p_x) \tag{27}$$

where p_x and p_y are the components of the robot's position on the field, according to the coordinate system convention introduced in Section 2. The vertical component ϕ_z is:

$$\phi_z = atan(v_z/v_r)_{t=t_f} \tag{28}$$

From (3), (5), and (8),

$$\begin{aligned} \phi_z &= atan((v_{zo} - g\ t_f)/v_{ro}) \\ &= atan((v_o \sin(\theta) - g\ t_f)/v_o \cos(\theta)) \\ &= \mp atan\left(\frac{\sqrt{{v_o}^2 \sin^2(\theta) - 2g(d_z - d_{zo})}}{v_o \cos(\theta)}\right) \end{aligned} \tag{29}$$

Since the desired result is $\cos(\phi)$, it is not necessary to solve for ϕ since its cosine is easier to obtain. One approach would be to take a unit vector e_x (pointing out from the target's backboard), and rotate it about the Z-axis by ϕ_r, and then about the Y-axis by ϕ_z, and measure the resulting component along the X direction:

$$\cos(\phi) = {e_x}^T R_y(\phi_z)\ R_z(\phi_r)\ e_x \tag{30}$$

where

$$R_y(\phi_z) = \begin{pmatrix} \cos(\phi_z) & 0 & \sin(\phi_z) \\ 0 & 1 & 0 \\ -\sin(\phi_z) & 0 & \cos(\phi_z) \end{pmatrix} \tag{31}$$

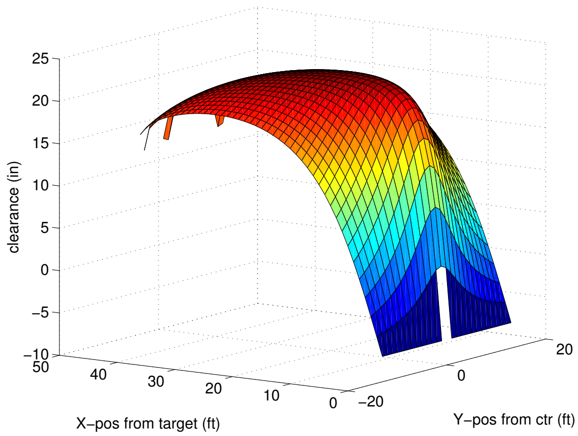

Figure 4: Shown are the clearances of the target's hole as the ball enters as a function of the robot's position on the floor.

$$R_z(\phi_r) = \begin{pmatrix} \cos(\phi_r) & -\sin(\phi_r) & 0 \\ \sin(\phi_r) & \cos(\phi_r) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (32)$$

Therefore, (30) becomes:

$$\cos(\phi) = \cos(\phi_r)\cos(\phi_z) \quad (33)$$

and this result is regardless of which rotation is performed first. Substituting

$$D_h = 30 \text{ in} = 2.5 \text{ ft} \quad (34)$$

$$D_b = 7 \text{ in} \approx 0.583 \text{ ft} \quad (35)$$

into (26), (33), (27), (30), and using the data plotted in Figure 3, the relationship between shooting clearance and the robot's position (p_x, p_y) on the floor is obtained. This is shown in Figures 4-5. Note that for the case of $(p_x = 27 \text{ ft}, p_y = 13 \text{ ft})$, i.e. the middle starting location of the autonomy round, the clearance is $\Delta D \approx 19.1$ inches, which is only slightly less than the maximum possible 23 inches.

This data can then be used in conjunction with empirical repeatability results from the launcher to obtain success probabilities versus the robot's position.

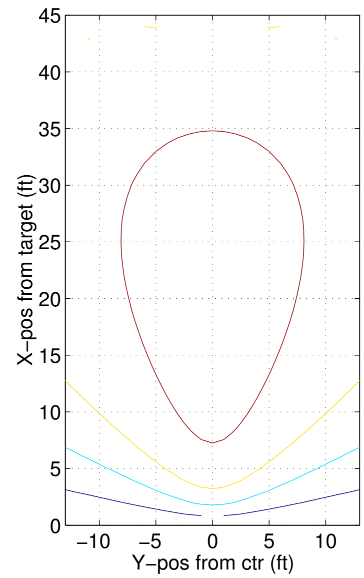

Figure 5: Shown are the contours of $\{0, 7, 14, 21\}$ ft (alternatively, $\{0, 1, 2, 3\}$ ball-diameter) clearances as a function of the robot's position on the floor.

8 Conclusions

This report has examined the problem of shooting a ball into a vertical target hole, using ideal kinematic equations. A number of interesting results have been obtained: First, given the parameters of the problem, the minimum launch angle θ for any configuration is $\approx 22.4°$. Second, the maximum horizontal range of the launcher is ≈ 44.54 ft. Third, the clearance between the target hole and the ball projected onto the entry plane is examined. This survey indicates that shooting from the corners of the court (say within 10 ft, from the one point goals) is not prudent. Furthermore, it sheds light into the feasibility of launching multiple simultaneous balls. Finally, during the autonomy round, if shooting from the starting position without moving at all, i.e. $(p_x = 27, p_y = 13)$ ft, the clearance is quite good (≈ 19.1 inches ≈ 2.7 ball-diameters), and the time in flight is ≈ 0.85 seconds.