A State-Space Model for a Differential Swerve

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1 Introduction

A differential swerve is an uncommon type of FRC drivetrain mechanism. Similar to swerve in its function, each differential swerve pod has a wheel and the ability to change the direction of the wheel (commonly referred to as the "azimuth" of the wheel.) The differential swerve is, however, different from regular swerve in its operation. Unlike a normal swerve the differential swerve's wheel velocity and azimuth is controlled by both motors, instead of each motor controlling either the wheel or azimuth. A good mechanical visualization of the differential swerve can be found here: `https://www.youtube.com/watch?v=Y1XtXkTUXsI`.

2 State Space Model Outline

First we will establish a general outline for our model as it fits into state-space form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{1}$$

We will describe the elements of our system that we want to control by defining our state vector. Let θ_a and ω_a be the angle and angular velocity, respectively, of the module, and let ω_w be the angular velocity of the wheel:

$$\mathbf{x} = \begin{bmatrix} \theta_a \\ \omega_a \\ \omega_w \end{bmatrix}$$

Then it's easy to find the derivative of the state vector, $\dot{\mathbf{x}}$:

$$\dot{\mathbf{x}} = \begin{bmatrix} \omega_a \\ \dot{\omega}_a \\ \dot{\omega}_w \end{bmatrix}$$

Next, we define our input vector. Let V_t and V_b be the voltage inputs of the two motors driving the differential swerve:

$$\mathbf{u} = \begin{bmatrix} V_t \\ V_b \end{bmatrix}$$

Finally, we will describe the parts of our system that we can directly measure by defining our measurement vector:

$$\mathbf{y} = \begin{bmatrix} \theta_a \\ \omega_w \end{bmatrix}$$

Now we need to find a set of **A**, **B**, **C**, and **D** matrices that satisfy the equations in (1). Finding **C** and **D** is easy:

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathbf{D} &= \mathbf{0} \end{aligned} \tag{2}$$

We can also figure out the skeleton of **A** and **B**:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 \\ ... & ... & ... \\ ... & ... & ... \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 0 & 0 \\ ... & ... \\ ... & ... \end{bmatrix} \end{aligned} \tag{3}$$

3 Model derivation

We need to find the equations describing the dynamics of the differential swerve in a form that we can use to fill in the missing parts of **A** and **B**.

We will first establish the basic forward kinematics equations for a differential swerve drive. Let ω_t be the angular velocity of the motor geared to the top main gear of the module, and let ω_b be the angular velocity of the motor geared to the bottom main gear of the module:

$$\omega_w = \frac{\omega_t + \omega_b}{2} \tag{4}$$

Let ω_a be the angular velocity of the azimuth of the module (i.e. the velocity at which the wheel rotates about a vertical axis):

$$\omega_a = \frac{\omega_t - \omega_b}{2} \tag{5}$$

These equations are *very* similar to the kinematics equations for a differential drive.

Unfortunately, we control only the voltage inputs to the two motors and not the output velocities. These voltage inputs are related to torque and angular velocity by the following equation [1, see eqn. 13.3]:

$$V = \frac{\tau}{K_t} R + \frac{\omega}{K_v} \tag{6}$$

In the above equation τ is torque, K_t is a constant relating the torque to the current through the motor (this relationship is assumed to be linear), R is a constant describing the internal resistance of the motor, ω is the angular velocity of the output shaft of the motor, and K_v is a constant relating the angular velocity to the back EMF of the motor. These constants can easily be calculated from a motor datasheet. There is also some gearing between the output shaft of the motor and the wheel or azimuth components of the swerve. This gearing will be denoted generally as G and will come into play later.

We would like to derive a relationship between the voltage of the top and bottom motors and the angular velocity of the wheel or the angular velocity of the azimuth, because it is easiest to design a controller if we have an equation in this form. We will start with our motor equation. τ_m and ω_m are the torque and angular velocity of either the top or bottom motor.

$$V = \frac{R}{K_t} \tau_m + \frac{1}{K_v} \omega_m \tag{7}$$

Now we will solve for τ_m.

$$\tau_m = \frac{K_t}{R} V - \frac{K_t}{K_v R} \omega_m \tag{8}$$

$G\tau_m = \tau_e$, where τ_e is the torque on some end effector; $\omega_m = G\omega_e$, where ω_e is the angular velocity of some end effector. We can substitute these gearing relationships into (8):

$$\begin{aligned} (\frac{\tau_e}{G}) &= \frac{K_t}{R} V - \frac{K_t}{K_v R} (G\omega_e) \\ \tau_e &= \frac{GK_t}{R} V - \frac{G^2 K_t}{K_v R} \omega_e \end{aligned} \tag{9}$$

We know that $\tau = J\dot{\omega}$, where J is the moment of inertia of the object the motor is rotating through its axis of rotation, and $\dot{\omega}$ is angular acceleration. We can substitute this into (9):

$$\begin{aligned} (J\dot{\omega}_e) &= \frac{GK_t}{R} V - \frac{G^2 K_t}{K_v R} \omega_e \\ \dot{\omega}_e &= \frac{GK_t}{RJ} V - \frac{G^2 K_t}{K_v R J} \omega_e \end{aligned} \tag{10}$$

We can now substitute (10) into the derivative of our two forward kinematics equations, (4) and (5) and change J to J_w and J_a, G to G_w and G_a, and V to V_t and V_b as applicable.

First, we will come up with our final dynamics equation for the wheel. Substituting into the wheel forward kinematics equation we get:

$$\dot{\omega}_w = -\frac{G_w K_t K_v V_b - G_w^2 K_t \omega_b}{2\, J_w K_v R} + \frac{G_w K_t K_v V_t - G_w^2 K_t \omega_t}{2\, J_w K_v R} \tag{11}$$

Simplify by collecting coefficients:

$$\dot{\omega}_w = \frac{G_w K_t}{2 J_w R}(V_t + V_b) - \frac{G_w^2 K_t}{2 J_w K_v R}(\omega_t + \omega_b) \tag{12}$$

Substitute ω_w for $\frac{\omega_t + \omega_b}{2}$:

$$\dot{\omega}_w = \frac{G_w K_t}{2 J_w R}(V_t + V_b) - \frac{G_w^2 K_t}{J_w K_v R}\omega_w \tag{13}$$

Next, we will come up with our final dynamics equation for the azimuth. Substituting into the azimuth forward kinematics equation we get:

$$\dot{\omega}_a = -\frac{G_a K_t K_v V_b - G_a^2 K_t \omega_b}{2\, J_a K_v R} + \frac{G_a K_t K_v V_t - G_a^2 K_t \omega_t}{2\, J_a K_v R} \tag{14}$$

Simplify by collecting coefficients:

$$\dot{\omega}_a = \frac{G_a K_t}{2 J_a R}(V_t - V_b) + \frac{G_a^2 K_t}{2 J_a K_v R}(\omega_t - \omega_b) \tag{15}$$

Substitute ω_a for $\frac{\omega_t - \omega_b}{2}$:

$$\dot{\omega}_a = \frac{G_a K_t}{2 J_a R}(V_t - V_b) - \frac{G_a^2 K_t}{J_a K_v R}\omega_a \tag{16}$$

4 Final Model

We will take the dynamics equations we just derived and fit them into state-space form. For convenience we will define the following constants:

$$\begin{aligned} C_w &= \frac{G_w K_t}{J_w R} \\ C_a &= \frac{G_a K_t}{J_a R} \end{aligned} \tag{17}$$

We can put our final wheel equation, (13), and azimuth equation, (16), into (3) to find **A** and **B**.

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -G_a \frac{C_a}{K_v} & 0 \\ 0 & 0 & -G_w \frac{C_w}{K_v} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 0 & 0 \\ \frac{1}{2} C_a & -\frac{1}{2} C_a \\ \frac{1}{2} C_w & \frac{1}{2} C_w \end{bmatrix} \end{aligned} \tag{18}$$

And we already know **C** and **D**:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathbf{D} = \mathbf{0} \tag{19}$$

References

[1] Tyler Veness. *Controls Engineering in the FIRST Robotics Competition.* 2020. [Online; accessed 2 June, 2020].