Trapezoidal-Exponential Motion Profiling

FRC Team 449 - The Blair Robot Project
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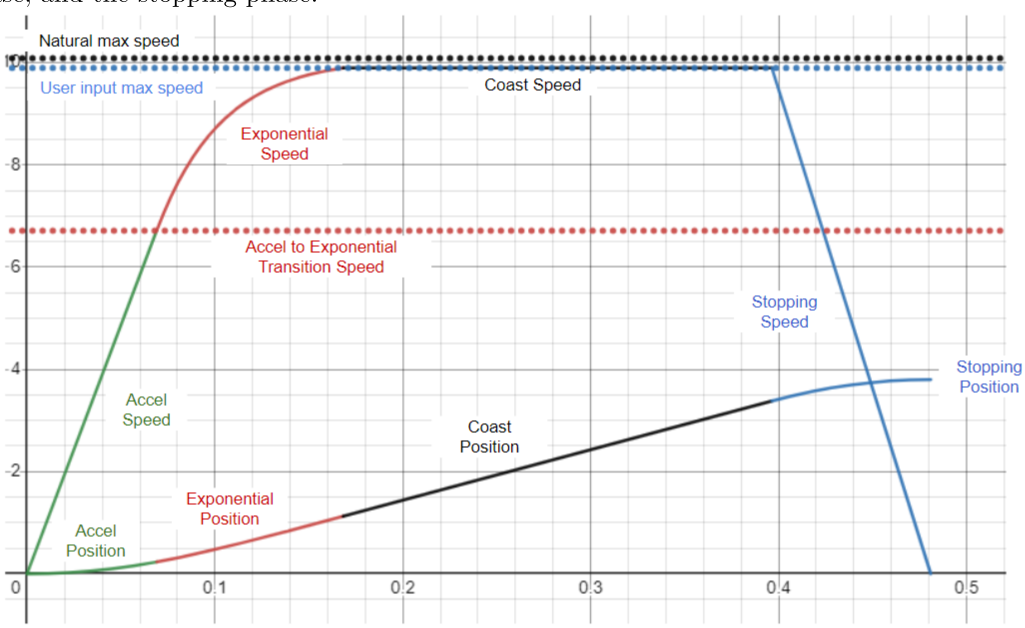
1 Introduction

Commonly used in controls, a trapezoidal motion profile is the least-time solution to a system whose motion is constrained by a constant max acceleration, a constant max speed, and a constant max deceleration. The use of this profile hinges on the assumption that the specified constant max acceleration is achievable up to and including the specified constant max speed, but this assumption does not necessarily hold. For example, FRC motors have a torque output which diminishes linearly to zero as their speed approaches their free speed, meaning that at some point prior to reaching their free speed, they will no longer be able to output the desired acceleration to stay on track with the motion profile. Past a certain point, their velocity exponentially converges towards their top speed.

In this paper I will derive the equations for the trapezoidal-exponential motion profile, an extended form of the trapezoidal motion profile which incorporates the exponential motor velocity behavior. This profile consists of up to four phases, shown in the image below: The constant acceleration phase, the exponential acceleration phase, the coasting phase, and the stopping phase.



2 Derivation

We begin by defining our profile inputs and constraints. Beginning at rest at position $x = 0$, we aim to arrive at position $x = x_f$ subject to the following constraints:

1. The speed-up acceleration is limited to $a = a_{lim}$.
2. The max speed of the system is limited to $v = v_{max}$.
3. The slow-down acceleration is limited to $a = -a_{stop}$
4. The system's natural max speed is $v = v_{free}$.
5. The speed at which the acceleration a_{lim} can no longer be sustained is at $v = v_{lim}$.

We will first solve the equations for the system's position and velocity as a function of Δt, the time since the system entered state i, given an initial position and velocity of $x_{i,0}$ and $v_{i,0}$. Throughout this derivation, v_{max} is assumed to be less than or equal to v_{free}.

2.1 Constant Acceleration Phase

During the first phase, we know the system is accelerating at a constant rate a_{lim} starting from $x_{1,0} = 0, v_{1,0} = 0$. This gives the position and velocity:

$$x_1(\Delta t_1) = \frac{1}{2}a_{lim}\Delta t_1^2$$
$$v_1(\Delta t_1) = a_{lim}\Delta t_1$$

2.2 Exponential Acceleration Phase

During the second phase, the system's velocity follows some exponential decay towards the system's top speed v_{free}. Since we don't know the shape of the exponential yet, we start by writing:

$$v_2(\Delta t_2) = v_{free} - Ae^{-B\Delta t_2}$$

Where A and B are some constants we don't know yet. What we do know, however, is that right at the moment we entered the exponential phase ($\Delta t_2 = 0$), our speed was $v_{2,0} = v_{lim}$ and our acceleration was a_{lim}. This means that:

$$v_2(0) = v_{free} - Ae^{-B\cdot 0}$$
$$v_{lim} = v_{free} - A$$
$$A = v_{free} - v_{lim}$$

By taking the derivative of the velocity, we find:

$$\frac{d}{d\Delta t_2}v_2(\Delta t_2) = 0 + ABe^{-B\Delta t_2}$$
$$a_2(\Delta t_2) = ABe^{-B\Delta t_2}$$
$$a_2(0) = ABe^{-B\cdot 0}$$
$$a_{lim} = AB$$

Plugging in our prior solution for A:

$$a_{lim} = (v_{free} - v_{lim})B$$

$$B = \frac{a_{lim}}{v_{free} - v_{lim}}$$

So we know our velocity in terms of Δt_2 is:

$$v_2(\Delta t_2) = v_{free} - (v_{free} - v_{lim})\exp(-\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_2)$$

Integrating to find position, we get:

$$x_2(\Delta t_2) = x_{2,0} + v_{free}\Delta t_2 + \frac{(v_{free} - v_{lim})^2}{a_{lim}}(\exp(-\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_2) - 1)$$

2.3 Coast Phase

During the third phase the system travels at a constant speed v_{max}. This gives position and velocity:

$$x_3(\Delta t_3) = x_{3,0} + v_{max}\Delta t_3$$

$$v_3(\Delta t_3) = v_{max}$$

2.4 Stopping Phase

During the fourth phase the system decelerates at a constant rate a_{stop}. This gives the position and velocity:

$$x_4(\Delta t_4) = x_{4,0} + v_{4,0}\Delta t_4 - \frac{1}{2}a_{stop}\Delta t_4^2$$

$$v_4(\Delta t_4) = v_{4,0} - a_{stop}\Delta t_4$$

Note that when $v(\Delta t_4) = 0$, $x(\Delta t_4) = x_f$.

2.5 Phase Transition Times

Now that we know how the system travels during each phase, we have to figure out at what time it goes from one phase to another. Note that depending on the system parameters, phases 2 and 3 might not be reached at all (though states 1 and 4 always occur). For example, if the system's $v_{max} < v_{lim}$, the system is always able to accelerate at its acceleration limit and so could skip the exponential acceleration phase entirely. Alternatively, if $v_{max} > v_{free}$, the system will never coast at v_{max} since it can never reach that speed, so the coast phase will never occur. In this way, phase 2, phase 3, or phases 2 and 3 may be skipped entirely.

We start by looking at the times when the constant acceleration phase transitions to each other phase. First, the transition time from phase 1 to phase 2 $\Delta t_{1,2}$ since the start of phase 1 can be found as:

$$v_1(\Delta t_{1,2}) = v_{lim}$$

$$a_{lim}\Delta t_{1,2} = v_{lim}$$

$$\Delta t_{1,2} = \frac{v_{lim}}{a_{lim}}$$

The transition time from phase 1 to phase 3 $\Delta t_{1,3}$ can be found as:

$$v_1(\Delta t_{1,3}) = v_{max}$$

$$a_{lim}\Delta t_{1,3} = v_{max}$$

$$\Delta t_{1,3} = \frac{v_{max}}{a_{lim}}$$

Finally, to find the transition time from phase 1 to phase 4 $\Delta t_{1,4}$, we first identify the following stop condition:

$$v_4(t_f)^2 = v_{4,0}^2 - 2a_{stop}d_{stop}$$

$$0 = v_{4,0}^2 - 2a_{stop}d_{stop}$$

$$d_{stop} = \frac{v_{4,0}^2}{2a_{stop}}$$

Where d_{stop} is the distance covered during the stopping phase. This means that we want to stop when we're at the position:

$$x_{4,0} = x_{stop} = x_f - d_{stop} = x_f - \frac{v_{4,0}^2}{2a_{stop}}$$

Since $x_{4,0} = x_1(\Delta t_{1,4})$ and $v_{4,0} = v_1(\Delta t_{1,4})$:

$$x_1(\Delta t_{1,4}) = x_f - \frac{v_1(\Delta t_{1,4})^2}{2a_{stop}}$$

$$\frac{1}{2}a_{lim}\Delta t_{1,4}^2 = x_f - \frac{(a_{lim}\Delta t_{1,4})^2}{2a_{stop}}$$

$$(\frac{a_{lim}}{2} + \frac{a_{lim}^2}{2a_{stop}})\Delta t_{1,4}^2 = x_f$$

$$\Delta t_{1,4} = \sqrt{\frac{x_f}{\frac{a_{lim}}{2} + \frac{a_{lim}^2}{2a_{stop}}}}$$

$$\Delta t_{1,4} = \sqrt{\frac{2x_f}{a_{lim} + a_{lim}^2/a_{stop}}}$$

Now we've solved all the phase transition times from the constant acceleration phase. Note that just because the time can be solved for doesn't mean that transition will actually occur. We will cover the logic for which transitions occur at a later point.

Next, we will find the phase transition times from the exponential acceleration phase to either the coasting phase or directly to the stopping phase.

The transition time from phase 2 to phase 3 $\Delta t_{2,3}$ can be found as:

$$v_2(\Delta t_{2,3}) = v_{max}$$

$$v_{free} - (v_{free} - v_{lim})\exp(-\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_{2,3}) = v_{max}$$

$$v_free - v_max = (v_{free} - v_{lim})\exp(-\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_{2,3})$$

$$\frac{v_{free} - v_{max}}{v_{free} - v_{lim}} = \exp(-\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_{2,3})$$

$$\ln\left(\frac{v_{free} - v_{max}}{v_{free} - v_{lim}}\right) = -\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_{2,3}$$

$$\Delta t_{2,3} = -\frac{v_{free} - v_{lim}}{a_{lim}}\ln\left(\frac{v_{free} - v_{max}}{v_{free} - v_{lim}}\right)$$

The transition time from phase 2 to phase 4 $\Delta t_{2,4}$ can be found using our previously solved stopping condition:

$$x_{4,0} = x_f - \frac{v_{4,0}^2}{2a_{stop}}$$

$$x_2(\Delta t_{2,4}) = x_f - \frac{v_2(\Delta t_{2,4})^2}{2a_{stop}}$$

$$x_{2,0} + v_{free}\Delta t_{2,4} + \frac{(v_{free} - v_{lim})^2}{a_{lim}}(\exp(-\frac{a_{lim}}{v_{free} - v_{lim}}\Delta t_{2,4}) - 1) = x_f - \frac{(v_{free} - (v_{free} - v_{lim})\exp(-\frac{a_{lim}}{v_{free}-v_{lim}}\Delta t_{2,4}))^2}{2a_{stop}}$$

Let $A = v_{free}, B = -(v_{free} - v_{lim}), C = -\frac{a_{lim}}{v_{free}-v_{lim}}, D = 2a_{stop}, \Delta x = x_f - x_{2,0}$:

$$A\Delta t_{2,4} + \frac{-(v_{free} - v_{lim})}{-\frac{a_{lim}}{v_{free}-v_{lim}}}(\exp(C\Delta t_{2,4}) - 1) = (x_f - x_{2,0}) - \frac{(A + B\exp(C\Delta t_{2,4}))^2}{D}$$

$$A\Delta t_{2,4} + \frac{B}{C}(\exp(C\Delta t_{2,4}) - 1) = \Delta x - \frac{(A + B\exp(C\Delta t_{2,4}))^2}{D}$$

$$AD\Delta t_{2,4} + \frac{BD}{C}(\exp(C\Delta t_{2,4}) - 1) = D\Delta x - (A^2 + 2AB\exp(C\Delta t_{2,4} + B^2\exp(2C\Delta t_{2,4}))$$

$$B^2\exp(2C\Delta t_{2,4}) + (\frac{BD}{C} + 2AB)\exp(C\Delta t_{2,4}) + AD\Delta t_{2,4} - \frac{BD}{C} - D\Delta x + A^2 = 0$$

As you may be able to guess, this equation cannot be solved analytically. Fortunately, it can be solved pretty easily using Newton's method to find the roots of the function:

$$f(\Delta t_{2,4}) = B^2\exp(2C\Delta t_{2,4}) + (\frac{BD}{C} + 2AB)\exp(C\Delta t_{2,4}) + AD\Delta t_{2,4} - \frac{BD}{C} - D\Delta x + A^2$$

For which we will also need the derivative:

$$f'(\Delta t_{2,4}) = 2B^2C\exp(2C\Delta t_{2,4}) + (BD + 2ABC)\exp(C\Delta t_{2,4}) + AD$$

Since this function was found from the intersection of a line with negative slope (the stopping phase) and an upwards decaying exponential (the exponential phase), it should only have one root. Additionally, that root is likely close to where the exponential phase began, so we can use an initial guess of $\Delta t_{2,4} = 0$.

The last phase transition we need to find is between the coast phase and the stopping phase, $\Delta t_{3,4}$. This can be found using the previously solved stopping condition:

$$x_{4,0} = x_f - \frac{v_{4,0}^2}{2a_{stop}}$$

$$x_3(\Delta t_{3,4}) = x_f - \frac{v_3(\Delta t_{3,4})^2}{2a_{stop}}$$

$$x_{3,0} + v_{max}\Delta t_{3,4} = x_f - \frac{v_{max}^2}{2a_{stop}}$$

$$\Delta t_{3,4} = \frac{x_f - x_{3,0}}{v_{max}} - \frac{v_{max}}{2a_{stop}}$$

We have now found expressions to solve for all possible phase transition times.

3 Phase Transition Logic

Now that we know how to find the times at which each transition could occur, we just have to put the pieces together. We always start in phase 1 constant acceleration. We can solve for our three transition times - $\Delta t_{1,2}$ which would land us in the exponential phase, $\Delta t_{1,3}$ which would land us in the coasting phase, and $\Delta t_{1,4}$ which would land us in the stopping phase.

We evaluate these three phase transition times and identify which one comes earliest, then perform the corresponding phase transition. For example, if $\Delta t_{1,3}$ is the smallest, our user imposed max speed is reached before the motor enters the exponential phase, so there's no need to enter the exponential phase and instead we can start coasting until it's time to stop. If $\Delta t_{1,4}$ is the smallest, our system needs to start stopping before it even reaches either its max speed or its exponential phase, so we skip right to the stopping phase.

Once we choose a time to exit the acceleration phase $\Delta t_{1,f}$, we want to solve the position and velocity when we switch so we can feed those into the next phase as the initial condition.

$$x_1(\Delta t_{1,f}) = \frac{1}{2} a_{lim} \Delta t_{1,f}^2$$

$$v_1(\Delta t_{1,f}) = a_{lim} \Delta t_{1,f}$$

If we end up in phase 2 we compute the two transition times $\Delta t_{2,3}$ and $\Delta t_{2,4}$ and identify which is earlier, then transition to the corresponding phase. Again, we want to solve the position and velocity at the end of phase 2 to feed into the initial condition of the next phase.

$$x_2(\Delta t_{2,f}) = x_{2,0} + v_{free} \Delta t_{2,f} + \frac{(v_{free} - v_{lim})^2}{a_{lim}} (\exp(-\frac{a_{lim}}{v_{free} - v_{lim}} \Delta t_{2,f}) - 1)$$

$$v_2(\Delta t_{2,f}) = v_{free} - (v_{free} - v_{lim}) \exp(-\frac{a_{lim}}{v_{free} - v_{lim}} \Delta t_{2,f})$$

If we end up in phase 3 we compute the transition time $\Delta t_{3,4}$ to find when we want to start stopping. Again, we want to solve the position and velocity at the end of the coasting phase to feed into the initial condition of the stopping phase.

$$x_3(\Delta t_{3,f}) = x_{3,0} + v_{max} \Delta t_{3,f}$$

$$v_3(\Delta t_{3,f}) = v_{max}$$

Once we are in the stopping phase, we find the time $\Delta t_{4,f}$ when the system arrives at the destination x_f at rest:

$$\Delta t_{4,f} = \frac{v_{4,0}}{a_{stop}}$$

To double check, we can solve for the final position to compare it against x_f:

$$x_4(\Delta t_{4,f}) = v_{4,0} \Delta t_{4,f} - \frac{1}{2} a_{stop} \Delta t_{4,f}^2$$

Finally, we can find our total time to target t_f by summing the various $\Delta t_{i,f}$ for i corresponding to each phase the system went through.

4 Solving in FRC Context

Now that we know how to solve the problem given the five parameters described earlier ($a_{lim}, v_{max}, a_{stop}, v_{free}, v_{lim}$), we want to know how to find what these parameter values should be. a_{lim}, v_{max}, and a_{stop} can be set through user preference. a_{lim} and a_{stop} can also be solved for from the current limits. Finally, v_{free} and v_{lim} must be solved for.

We start with a_{lim}. Consider a system with mass m, gearing G:1, wheel or pulley radius r, subject to a constant external force g (which would be 0 in the case of a drive train), and powered by a set of motors with combined stall torque τ_{stall}, individual stall current I_{stall}, individual free current I_{free}, and free speed ω_{free} (all in SI units). We subject the system to a stator current limit of I_{lim}. We know:

$$a_{lim} = \frac{F_{lim}}{m} - g$$

Where F_{lim} is the force applied at the wheel/pulley at the current limit.

$$a_{lim} = \frac{\tau_{lim} G/r}{m} - g$$

Where τ_{lim} is the torque applied at the current limit.

$$a_{lim} = \frac{\left(\frac{I_{lim}-I_{free}}{I_{stall}-I_{free}}\right)\tau_{stall} G}{mr} - g$$

To match the net weight "felt" by the system in both acceleration and deceleration, we can find a_{stop} by reversing the gravity in a_{lim}:

$$a_{stop} = a_{lim} + 2g$$

v_{lim} is the speed at which the motors stop being able to produce an acceleration of a_{lim}. To find this point, we solve what maximum acceleration a_{max} the motors can produce in terms of their current speed ω:

$$a_{max}(\omega) = \frac{F_m ax(\omega)}{m} - g = \frac{\tau_{max}(\omega) G}{mr} - g = \frac{\tau_{stall}(1 - \omega/\omega_{free})G}{mr} - g$$

We can set this equal to a_{lim} to find ω_{lim}, past which the motor can no longer accelerate at a_{lim}:

$$a_{max}(\omega_{lim}) = a_{lim}$$

$$\frac{\tau_{stall}(1 - \omega_{lim}/\omega_{free})G}{mr} - g = \frac{\left(\frac{I_{lim}-I_{free}}{I_{stall}-I_{free}}\right)\tau_{stall} G}{mr} - g$$

$$(1 - \omega_{lim}/\omega_{free}) = \left(\frac{I_{lim} - I_{free}}{I_{stall} - I_{free}}\right)$$

$$\omega_{lim} = \omega_{free}\left(1 - \left(\frac{I_{lim} - I_{free}}{I_{stall} - I_{free}}\right)\right)$$

We now find v_{lim} as:

$$v_{lim} = \omega_{lim}\frac{2\pi r}{G} = \frac{2\pi r \omega_{free}}{G}\left(1 - \left(\frac{I_{lim} - I_{free}}{I_{stall} - I_{free}}\right)\right)$$

We can also find v_{free}, the system's natural max speed, as the point where $a_{max} = 0$:

$$a_{max}(v_{free}G/(2\pi r)) = 0$$
$$\frac{\tau_{stall}(1 - v_{free}G/(2\pi r)/\omega_{free})G}{mr} - g = 0$$
$$1 - v_{free}G/(2\pi r)/\omega_{free} = \frac{mgr}{G\tau_{stall}}$$
$$v_{free} = \frac{2\pi r\omega_{free}}{G}\left(1 - \frac{mgr}{G\tau_{stall}}\right)$$

A few notes:

1. For a drivetrain, set $g = 0$.
2. For a tilted elevator at an angle θ from horizontal, set $g = 9.81\sin(\theta)$. Note that friction increases as θ deviates from 90°, so additional load may be desirable to compensate.
3. For a cascade elevator with n stages and a gearbox geared $G_0 : 1$, the value of G in the equations should be $G = G_0/n$, as the cascade acts as a gear speed-up ratio. Do not increase the mass m as is sometimes done to simulate the cascade action - this G correction covers that.
4. Torque efficiency η can be incorporated by multiplying the initial motor stall torque $\tau_{stall,0}$ as $\tau_{stall} = \eta\tau_{stall,0}$.
5. If you have n motors, set $\tau_{stall} = n\tau_{stall,0}$. None of the other motor parameters need to change.